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A theory for wave-power absorption by two independently oscillating bodies

By M. A. SROKOSZ AND D. V. EVANS
Department of Mathematics, University of Bristol, England

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In a recent paper (Evans 1976) a theory was presented for the behaviour of an oscillating two-dimensional cylinder of any shape which was capable of absorbing energy from a given regular sinusoidal wave. In particular an expression was derived for the efficiency of power absorption of the cylinder when oscillating in a single mode in terms of properties of the solution of the so-called radiation problem in which the cylinder is *forced* to oscillate in the appropriate mode in the absence of the incident wave train.

In the present paper this theory is extended to *two* independent cylinders of arbitrary shape each oscillating in a single mode and capable of absorbing energy in that mode. A general expression for the efficiency is derived which depends on properties of the solution to a new radiation problem, in which one cylinder is forced to oscillate in the presence of the other cylinder, which is held fixed in its equilibrium position. In this case, the efficiency also depends on cross-coupling coefficients related to the force on the fixed cylinder due to the motion of the oscillating cylinder.

It is shown that the cylinders can be tuned to absorb *all* the incident wave energy at a given frequency even for symmetric cylinders, in contrast to the single symmetric cylinder, for which the maximum efficiency has been shown to be 50 %.

The general solution to the new radiation problem is derived in terms of the solution to the radiation problem for a single cylinder, by assuming that the cylinders are far enough apart for local wave effects to be negligible.

The special case of two widely spaced rolling vertical plates is considered in detail and curves showing the variation of efficiency with wavelength are given for a variety of plate spacings and points of rotation.

1. Introduction

There is considerable interest at present in devising methods for extracting energy from ocean waves and a number of research groups are actively developing their own particular device (see, for example, Kenward 1976). In parallel with this experimental work a number of theoretical papers on wave energy have appeared. Thus Evans (1976) derived a general theory for the efficiency of wave absorption of a long cylinder of arbitrary cross-section oscillating in a single mode such as heave, surge or roll, or, in some cases, in a combination of two modes. Evans also derived results for the efficiency of a single three-dimensional body with a vertical axis of symmetry oscillating in one or more modes. In each case the expression for the efficiency required knowledge of the added mass and damping coefficients for the body or cylinder as functions of wave frequency and also the amplitude of the waves at large distances from the cylinder due to forced motion of the cylinder in the wave-absorbing mode. Some of the results derived in Evans (1976) were also obtained simultaneously by Mei (1976) and Newman (1976).

Count (1978) generalized Evans' results to include an arbitrary cylinder oscillating in more than one mode and an articulated cylinder, in order to model the behaviour of the Salter 'duck' device and its backbone and also the Cockerell contouring rafts device. Budal, in a series of papers (Budal & Falnes 1975*a*, *b*, *c*; Budal 1977), has concentrated on predicting the maximum absorption capability of various configurations of three-dimensional devices. Recently Standing (1978) has developed a computer program capable of determining hydrodynamic coefficients for a wide class of two- and three-dimensional problems and has applied it to predicting the performance of both the Salter duck and the NEL oscillating water column device in three-dimensional regular waves.

One of the most important conclusions reached by Evans (1976), Mei (1976) and Newman (1976) was that a cylinder which is capable of absorbing energy in two or more distinct modes of oscillation can absorb *all* the energy in a given incident wave of small amplitude. This result forms the basis for the submerged circular cylinder wave-energy device and has been confirmed experimentally (Evans *et al.* 1979).

In the present work we consider the problem of two arbitrary cylinders oscillating independently and capable of absorbing energy in a single mode from a given incident wave. There is no difficulty in principle in generalizing to any number of cylinders oscillating in two or three modes but in practice the analysis becomes unwieldy.

The problem is formulated in § 2 in terms of the linearized equations of motion and boundary conditions. It is shown how the general problem can be regarded as the superposition of the solution to the scattering problem in which the cylinders are held fixed in the incident wave, and the solution to the radiation problem in which each cylinder makes forced oscillations in turn, the other cylinder being held fixed, in the absence of the incident wave.

In § 3 it is shown, by considering the wave field far from the cylinders, that the maximum efficiency of wave-energy absorption is 100 % and the required displacements of the cylinders for this to occur are determined.

The equations of motion of the cylinders are derived in § 4 under the assumption that each cylinder can absorb energy through a simple linear spring–damper system, each damper absorbing a net amount of energy over a period. An alternative expression for the efficiency to that derived in § 3 is obtained by considering the net work being done on the cylinders. The displacements of the cylinders for maximum efficiency are then used to determine the values of the spring and damper constants which ensure that all the energy is absorbed from an incident wave of a given frequency. The system is then said to be 'tuned' to that frequency. A verification that, with these values of the spring and damper constants, the dampers do indeed absorb all the energy in the incident wave at the tuning frequency is given in appendix B.

In § 5 an approximate method is presented for solving the radiation and scattering problems for two arbitrary cylinders in terms of the solution of those problems for a single cylinder. The method, first used by Ohkusu (1974), is based on the assumption that the cylinders are spaced far enough apart for the local wave field in the vicinity of one cylinder not to influence the other cylinder. The only interaction between the cylinders is due to the propagating-wave terms which occur in the radiation and scattering problems for single cylinders. Mathematically, the approximation is equivalent to the assumption that the wavelength is small compared with the distance between the cylinders but it has been shown by Ohkusu (1974) that the method is valid over a much wider range of the ratio of wavelength to cylinder spacing. Expressions are derived for generalized added-mass and damping coefficients for two cylinders, being the forces on each cylinder due to the motion of one cylinder in a given mode, the other being held fixed. Also given are the wave amplitudes at large distances from the cylinders, and the reflexion and transmission coefficients when both cylinders are held fixed in a given incident wave. In each case the expressions depend on the solution to the corresponding problem for a single cylinder.

The special case of two vertical thin barriers is considered in §§ 6 and 7. This enables known explicit results derived by Ursell (1947, 1948) for a single barrier to be used and these are presented in § 6.

In § 7 the results for the hydrodynamic coefficients for two vertical thin barriers are presented and discussed. A check on the widely spaced approximation to the reflexion coefficient is made by comparison with more accurate results for that problem obtained by Evans & Morris (1972). It is shown that the approximation is valid even when the wavelength is *larger* than the barrier spacing, suggesting that the method can provide good results for the other hydrodynamic coefficients where no check is available.

Curves showing the variation of efficiency of wave-energy absorption with wave frequency for different barrier spacings and points of rotation are also presented and discussed in § 7.

New relations between the various hydrodynamic coefficients for a system of independent two-dimensional cylinders are given in appendix A. These generalize results such as the Haskind & Newman relations for a single cylinder (Newman 1976).

2. Formulation

We consider the motion of two cylinders of arbitrary cross-section, situated either on or beneath the free surface, each moving independently in one mode and capable of extracting power in that mode from the incident wave. We assume that the cylinders are long, with horizontal generators, and that the fluid motion is purely two-dimensional, being confined to planes normal to the cylinder generators. Here the generators of one cylinder are parallel to those of the other.

Cartesian co-ordinates (x, y) are chosen such that $y = 0$ is the undisturbed free surface, with y vertically upwards and x to the right. Under the usual assumptions of linearized water wave theory a velocity potential $\Phi(x, y, t)$ exists which satisfies

$$\nabla^2\Phi = 0 \quad \text{in the fluid,} \tag{2.1}$$

$$\partial^2\Phi/\partial t^2 + g\,\partial\Phi/dy = 0 \quad \text{on} \quad y = 0. \tag{2.2}$$

It is assumed that a small amplitude sinusoidal wave train of frequency ω is incident from $x = +\infty$ upon the cylinders, and that the generators of the cylinders are parallel to the wave crests. Each cylinder is constrained to make small amplitude oscillations in response to the incident wave, in one mode only (either sway, heave or roll but not a combination of these). In the absence of waves it is assumed that each cylinder is held in equilibrium by a combination of buoyancy forces and a spring-and-damper system connected to each cylinder. This system is capable of extracting power and it will be assumed that the power take-off mechanism is described by a single linear damper (for each cylinder) with a resistance to motion which is proportional to velocity.

On each cylinder we impose the condition that the component of cylinder velocity normal to the cylinder is equal to the normal velocity of the fluid at that point. Let $\zeta_{ij}(t)$ describe the displacement of the ith cylinder ($i = 1, 2$) from its equilibrium position. Here $j = 1$ relates to sway motions, $j = 2$ to heave motions and $j = 3$ to roll motions. The linearized conditions on the equilibrium positions of the cylinders are then

$$\partial\Phi/\partial n_i = \dot{\zeta}_{ij} n_{ij} \quad (i = 1, 2; \quad j = 1, 2, 3) \tag{2.3}$$

for (x, y) on S_i, the wetted surface of the ith cylinder, where $\mathbf{n}_i = (n_{i1}, n_{i2})$ is the normal from the ith cylinder into the fluid at the point (x, y) and $n_{i3} = n_{i2}(x-\alpha_i)+n_{i1}(y-\beta_i)$, where (α_i, β_i) is the point of rotation of the ith body.

It is convenient to eliminate the assumed harmonic time dependence by writing

$$\Phi(x, y, t) = \mathrm{Re}\{\phi(x, y)\, e^{i\omega t}\}, \tag{2.4}$$

where ω is the radian frequency.

Now the complex-valued time-independent potential $\phi(x, y)$ may be written

$$\phi(x, y) = \omega^{-1}gA\phi_s + i\omega\xi_{1j}\phi_{1j} + i\omega\xi_{2k}\phi_{2k}, \tag{2.5}$$

where
$$\zeta_{ij} = \mathrm{Re}\{\xi_{ij}\, e^{i\omega t}\} \quad (i = 1, 2; \quad j, k = 1, 2, 3) \tag{2.6}$$

and A is a complex constant. Here the complex potential ϕ_s is the solution of the scattering problem in which both cylinders are held fixed in an incident wave from $x = +\infty$ of unit amplitude potential. The complex potential ϕ_{1j} is the solution of the radiation problem in which a normal velocity $\mathrm{Re}\{n_{1j}\, e^{i\omega t}\}$ corresponding to small oscillations of unit amplitude in the jth mode ($j = 1, 2$ or 3) is prescribed on the first cylinder and the second cylinder is held fixed. Similarly ϕ_{2k} describes small oscillations of unit amplitude of the second cylinder in the kth mode ($k = 1, 2, 3$) with the first cylinder held fixed.

Then (2.3) is satisfied since

$$\partial\phi_s/\partial n_p = 0, \quad \partial\phi_{ij}/\partial n_p = n_{pj}\delta_{ip} \quad \text{on} \quad S_p \quad (i, p = 1, 2; \quad j = 1, 2, 3), \tag{2.7}$$

where
$$\delta_{ip} = \begin{cases} 0 & \text{if} \quad i \neq p, \\ 1 & \text{if} \quad i = p. \end{cases}$$

The wave elevation is given by

$$g^{-1}\,\partial\Phi(x, 0, t)/\partial t = \mathrm{Re}\{i\omega g^{-1}\,\phi(x, 0)\, e^{i\omega t}\},$$

so that the incident wave has amplitude A if we assume

$$\phi_s \sim \begin{cases} (e^{iKx} + R\, e^{-iKx})\, e^{Ky} & \text{as} \quad x \to +\infty, & (2.8) \\ T\, e^{iKx+Ky} & \text{as} \quad x \to -\infty, & (2.9) \end{cases}$$

where R and T are the complex reflexion and transmission coefficients for the scattering problem. Here $K = \omega^2/g$. For ϕ_{ij} we assume the following behaviour as $x \to \pm\infty$:

$$\phi_{ij} \sim A_{ij}^{\pm} e^{\mp iKx+Ky} \quad \text{as} \quad x \to \pm\infty. \tag{2.10}$$

It is possible, by using Green's theorem, to derive relations between the various quantities associated with the scattering and radiation problems and these are derived in appendix A. The relations are generalizations of the Haskind, Newman and other relations (see Newman 1976) to the case of more than one body. In the following sections these results are used to consider the efficiency of power extraction by two independently oscillating cylinders.

3. The maximum efficiency of power absorption

The efficiency of the system will be defined as the proportion of the available power per unit frontage of the incident wave which is extracted by the two cylinders. This will clearly depend on the details of the coupling between the cylinders and the fluid. However, it is possible to obtain some information about the maximum efficiency without knowing the details of the coupling.

From (2.5) and (2.8)–(2.10) we obtain

$$\phi \sim \begin{cases} (\omega^{-1}gA)\,(e^{iKx}+R_1\,e^{-iKx})\,e^{Ky} & \text{as} \quad x \to +\infty, & (3.1) \\ (\omega^{-1}gA)\,T_1\,e^{iKx+Ky} & \text{as} \quad x \to -\infty, & (3.2) \end{cases}$$

where
$$R_1 = R + iKA^{-1}(\xi_{1j}A^+_{1j} + \xi_{2k}A^+_{2k}), \tag{3.3}$$

$$T_1 = T + iKA^{-1}(\xi_{1j}A^-_{1j} + \xi_{2k}A^-_{2k}). \tag{3.4}$$

Following Evans (1976), the efficiency E of the system is just

$$E = 1 - R_1\bar{R}_1 - T_1\bar{T}_1 \tag{3.5}$$

(here an overbar denotes a complex conjugate). As $R_1\bar{R}_1$, $T_1\bar{T}_1 \geqslant 0$, it is clear from (3.5) that the maximum value of E will be 1, when $R_1 = T_1 = 0$. To find when this is attained we set $R_1 = T_1 = 0$ in (3.3) and (3.4), and then solve for ξ_{1j} and ξ_{2k}. If we now use equations (A 7) and (A 8) from appendix A to replace R and T by expressions involving $A^{\pm}_{ij}$, we obtain after some manipulation

$$\xi_{1j} = (iK)^{-1}A\overline{A^-_{2k}}(\overline{A^+_{1j}}\,\overline{A^-_{2k}} - \overline{A^-_{1j}}\,\overline{A^+_{2k}})^{-1}, \tag{3.6}$$

$$\xi_{2k} = -(iK)^{-1}A\overline{A^-_{1j}}(\overline{A^+_{1j}}\,\overline{A^-_{2k}} - \overline{A^-_{1j}}\,\overline{A^+_{2k}})^{-1} \tag{3.7}$$

as the conditions for complete absorption. With these values of ξ_{1j} and ξ_{2k} we have $E_{\max} = 1$, so that all the energy in the incident wave is absorbed.

This result could have been anticipated by the following argument. Suppose that two cylinders are symmetrically positioned with respect to the line $x = 0$. If both cylinders make identical heave oscillations in phase, the waves radiated to $x = \pm\infty$ will have the same amplitude and phase. If the cylinders heave exactly out of phase the waves radiated to $x = \pm\infty$ will have the same amplitude but opposite phase. It follows that there exists a linear combination of these two heave motions which will cancel the waves at, say, $x = -\infty$ while doubling the wave amplitude at $x = +\infty$. By reversing the time co-ordinate, we see that there exists a heave motion of the two cylinders which completely absorbs a given incident wave from $x = +\infty$. The same argument applies to roll and sway motions. Notice that the above argument also holds if each cylinder is symmetric about a vertical axis through its equilibrium position.

Now it has been shown (Evans 1976) that a single such cylinder can absorb a maximum of 50 % of the incident wave energy, 25 % being reflected and 25 % transmitted. It could be argued that a second symmetric cylinder can absorb at most 50 % of what is transmitted past the first cylinder, 25 % being reflected back to the first cylinder, which in turn absorbs 50 % of this energy and so on. This leads to a maximum of $\frac{2}{3}$ of the incident energy being absorbed with $\frac{4}{15}$ reflected and $\frac{1}{15}$ transmitted. This argument is unsound not least because no allowance for favourable phase cancellation has been made, and as we have seen, 100 % absorption is in fact possible.

It is of interest to note that, if we drop the subscripts 1 and 2 and consider this formulation as representing the motion of one cylinder in two distinct modes, we have also shown that any cylinder moving in two distinct modes can extract all the energy in an incident wave (Evans 1976). In fact by suitably altering the notation all the results of §§ 2–4 and appendices A and B can be shown to apply to one cylinder in two modes.

4. The equations of motion of the two cylinders

We now consider in detail the motion of the two cylinders. We shall assume that each cylinder's motion is resisted by mechanical forces which can be modelled by a simple spring-and-damper system. Thus $\zeta_{ij}(t)$ satisfies

$$m_i \ddot{\zeta}_{ij} = -d_i \dot{\zeta}_{ij} - k_i \zeta_{ij} + F_{ij} \quad (i = 1, 2; \quad j = 1, 2, 3), \tag{4.1}$$

where d_i and k_i are the damper and spring constants and m_i the mass (or moment of inertia, if $j = 3$) of the ith cylinder. For heave and roll motion ($j = 2, 3$), k_i may also include a buoyancy force. The terms $d_i \dot{\zeta}_{ij}$ ($i = 1, 2$) allow a net amount of work to be done on the cylinders over a period, provided $d_i \neq 0$ for either $i = 1$ or $i = 2$. The term F_{ij} is the total hydrodynamic 'force' on the ith cylinder. For $j = 1, 2$ the force is horizontal and vertical respectively; F_{i3} is the moment about the point of rotation.

The total hydrodynamic forces can be conveniently separated into two parts. We write

$$F_{ij} = F^s_{ij} + F^r_{ij} \quad (i = 1, 2; \quad j = 1, 2, 3), \tag{4.2}$$

where F^s_{ij} is the force acting on the ith cylinder in the jth direction when both cylinders are assumed to be held fixed in the presence of the incident wave. F^r_{ij} is the force on the ith cylinder in the jth direction due to its own motion and the motion of the other cylinder in the absence of the incident wave. From (A 16) we obtain

$$F^r_{1j} = -M_{1j1j} \ddot{\zeta}_{1j} - B_{1j1j} \dot{\zeta}_{1j} - M_{1j2k} \ddot{\zeta}_{2k} - B_{1j2k} \dot{\zeta}_{2k} \tag{4.3}$$

and

$$F^r_{2k} = -M_{2k2k} \ddot{\zeta}_{2k} - B_{2k2k} \dot{\zeta}_{2k} - M_{2k1j} \ddot{\zeta}_{1j} - B_{2k1j} \dot{\zeta}_{1j}, \tag{4.4}$$

where M_{ijpk} and B_{ijpk} ($i, p = 1, 2$) are generalized added-mass and damping coefficients, as defined in (A 17). Furthermore, from (A 14), (A 19) and (A 20)

$$F^s_{ij} = \mathrm{Re}\{\rho g A A^+_{ij} e^{i\omega t}\}, \tag{4.5}$$

$$B_{ijpk} = \tfrac{1}{2}\rho\omega[\overline{A^+_{ij}} A^+_{pk} + \overline{A^-_{ij}} A^-_{pk}] = \tfrac{1}{2}\rho\omega[A^+_{ij} \overline{A^+_{pk}} + A^-_{ij} \overline{A^-_{pk}}]. \tag{4.6a, b}$$

It follows from (4.1)–(4.5) and (2.6) that

$$(Z_{1j1j} + d_1)\, \xi_{1j} = (i\omega)^{-1} \rho g A A^+_{ij} - Z_{1j2k} \xi_{2k}, \tag{4.7}$$

$$(Z_{2k2k} + d_2)\, \xi_{2k} = (i\omega)^{-1} \rho g A A^+_{2k} - Z_{1j2k} \xi_{1j}, \tag{4.8}$$

where $$Z_{ijij} = B_{ijij} + i\omega(m_i + M_{ijij}) - i\omega^{-1}k_i \quad (i = 1, 2), \tag{4.9}$$

$$Z_{1j2k} = B_{1j2k} + i\omega M_{1j2k}. \tag{4.10}$$

Use has been made of the symmetry relations (A 18).

For E to be equal to 1 we require ξ_{1j} and ξ_{2k} to satisfy (3.6) and (3.7) respectively. We therefore substitute these values into (4.7) and (4.8) to find the values of d_i and k_i ($i = 1, 2$) which give maximum power absorption. Thus we have

$$\begin{aligned} Z_{1j1j} + d_1 &= \rho\omega(A^+_{ij}/\overline{A^-_{2k}})\,[\overline{A^+_{1j}}\,\overline{A^-_{2k}} - \overline{A^-_{1j}}\,\overline{A^+_{2k}}] + Z_{1j2k}\overline{A^-_{1j}}/\overline{A^-_{2k}} \\ &= \rho\omega[|A^+_{1j}|^2 + |A^-_{1j}|^2] - \{\rho\omega[A^+_{1j}\,\overline{A^+_{2k}} + A^-_{1j}\,\overline{A^-_{2k}}] - Z_{1j2k}\}\,(\overline{A^-_{1j}}/\overline{A^-_{2k}}) \\ &= 2B_{1j1j} + \{Z_{1j2k} - 2B_{1j2k}\}\,(\overline{A^-_{1j}}/\overline{A^-_{2k}}), \end{aligned}$$

after use of (4.6). Hence, equating real and imaginary parts, we obtain

$$k_1 = \omega^2(m_1 + M_{1j1j}) - \omega\,\mathrm{Im}\,\{(i\omega M_{1j2k} - B_{1j2k})\,\overline{A^-_{1j}}/\overline{A^-_{2k}}\}, \tag{4.11}$$

$$d_1 = B_{1j1j} + \mathrm{Re}\,\{(i\omega M_{1j2k} - B_{1j2k})\,\overline{A^-_{1j}}/\overline{A^-_{2k}}\}. \tag{4.12}$$

Similarly we obtain

$$k_2 = \omega^2(m_2 + M_{2k2k}) - \omega\,\mathrm{Im}\,\{(i\omega M_{1j2k} - B_{1j2k})\,\overline{A^-_{2k}}/\overline{A^-_{1j}}\}, \tag{4.13}$$

$$d_2 = B_{2k2k} + \mathrm{Re}\,\{(i\omega M_{1j2k} - B_{1j2k})\,\overline{A^-_{2k}}/\overline{A^-_{1j}}\}. \tag{4.14}$$

These values of d_i and k_i ($i = 1, 2$) give $E = 1$ at the frequency ω.

Notice that, in terms of the mean work being done on the cylinders by the hydropynamic forces, the efficiency E can be written

$$\begin{aligned} E &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \{\dot{\zeta}_{1j}F_{1j} + \dot{\zeta}_{2k}F_{2k}\}\,dt/[\rho g^2|A|^2(4\omega)^{-1}] \\ &= (\rho g^2|A|^2)^{-1}\,2\omega^2\{d_1\,|\xi_{1j}|^2 + d_2\,|\xi_{2k}|^2\}, \end{aligned} \tag{4.15}$$

using (4.1) and (2.6). On substitution of the values of ξ_{1j}, ξ_{2k}, d_1 and d_2 given by (3.6), (3.7), (4.12) and (4.14) we obtain $E = 1$, in agreement with our previous derivation in § 3. The details of this calculation may be found in appendix B.

As noted earlier, if we drop the subscripts 1 and 2, this result can be applied to a single body in two modes. In particular, for a circular cylinder we have from symmetry $M_{jk} = 0$ and $B_{jk} = 0$ ($j \neq k$) for $j = 1$ (heave) and $k = 2$ (sway). Hence from (4.11)–(4.14) we obtain the same conditions on d_i and k_i to achieve $E = 1$ as in Evans [1976, equation (8.6)].

Note that the quantities M_{ijpk}, B_{ijpk} and $A^{\pm}_{ij}$ are all frequency dependent. Thus to achieve $E_{\max} = 1$ at $\omega = \omega_0$ we require the values of d_i and k_i given by (4.11)–(4.14) to be satisfied at $\omega = \omega_0$. If these values of d_i and k_i for $\omega = \omega_0$ are substituted into (4.15) we obtain $E = E(\omega, \omega_0)$, a function of ω and ω_0, and in particular $E(\omega_0, \omega_0) = 1$. The system is then said to be 'tuned' to $\omega = \omega_0$, so that E attains its maximum at $\omega = \omega_0$.

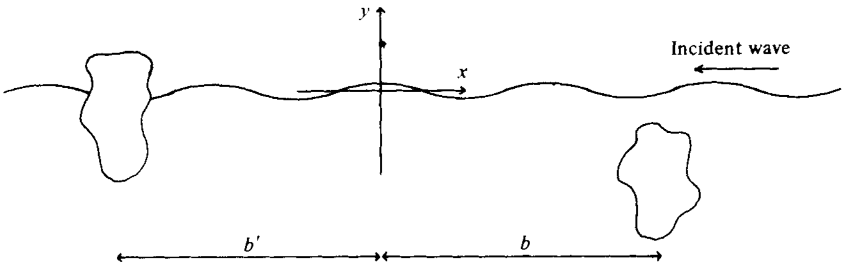


FIGURE 1. Configuration of two cylinders for wide-spacing approximation.

It is of both theoretical and practical interest to study the variation of $E(\omega, \omega_0)$ with ω, and to do this we substitute ξ_{1j} and ξ_{2k} from (4.7) and (4.8) into (4.15) to obtain

$$E(\omega, \omega_0) = 2\rho\omega\,|(Z_{1j1j}+d_1)(Z_{2k2k}+d_2)-Z^2_{1j2k}|^{-2}\{d_2\,|(Z_{1j1j}+d_1)A^+_{2k}-Z_{1j2k}A^+_{1j}|^2 + d_1\,|(Z_{2k2k}+d_2)A^+_{1j}-Z_{1j2k}A^+_{2k}|^2\}, \quad (4.16)$$

where d_i and k_i are given by (4.11)–(4.14) with $\omega = \omega_0$.

Clearly (4.16) gives E as a complicated function of ω and ω_0 and does not yield much information on the variation of E with ω. It is necessary therefore to solve the radiation problem for a particular configuration of two bodies and then to use the values of $M_{ijpk}(\omega)$, $B_{ijpk}(\omega)$ and $A^{\pm}_{ij}(\omega)$ given by that solution to see how E varies with ω in that particular case.

Few exact solutions are known to the authors for the radiation problem of one body in the presence of another. Those that are known deal with symmetrically placed bodies moving in unison: a catamaran-type configuration (e.g. Wang & Wahab 1971; Wang 1970). For the radiation problem involving one body fixed and one moving, these solutions are of no use in determining added-mass and damping coefficients or the amplitude of waves at infinity. Therefore, in the remainder of this paper, a good approximate method is developed in which the solution of the two-body problem is found in terms of the solutions to the scattering and radiation problems for a single body. The particular case of two surface-piercing, vertical, thin barriers is studied in detail and the variation of $E(\omega, \omega_0)$ with ω considered.

5. Wide-spacing approximation

In this section approximate solutions to the radiation and scattering problems are derived on the basis of the assumption that the distance between the two cylinders is large compared with the wavelength. This assumption enables us to neglect the interaction due to local effects, which decay with distance from each cylinder, and to assume that the only interaction that takes place is due to the travelling waves which pass between the two cylinders. This type of approximation has been used by other authors (Newman 1977; Ohkusu 1974) to examine similar problems. Under the wide-spacing assumption the solution in the neighbourhood of each cylinder can be written as a combination of the solutions to the scattering and radiation problems for that cylinder in the absence of the other.

We define time-independent potentials $\psi^r_{ij}(x,y)$ and $\psi^s_{i\pm}(x,y)$ to be the solutions of the radiation (in the jth mode) and scattering problems for the ith cylinder in the absence of the other cylinder. In each case ($i = 1, 2$) the cylinder is taken to be situated at the origin. The $\pm$ sign in $\psi^s_{i\pm}$ denotes waves incident from $x = \pm\infty$ in the scattering problem. Then ψ^r_{ij} and $\psi^s_{i\pm}$ satisfy Laplace's equation in the fluid, the free-surface condition, together with the normalized conditions

$$\partial\psi^s_{i\pm}/\partial n_i = 0 \quad \text{on} \quad S_i \quad (i = 1, 2), \tag{5.1}$$

$$\partial\psi^r_{ij}/\partial n_i = n_{ij} \quad \text{on} \quad S_i \quad (i = 1, 2; \quad j = 1, 2, 3). \tag{5.2}$$

Also, we assume that

$$\psi^r_{ij} \sim a^{\pm}_{ij} e^{\mp iKx+Ky} \quad \text{as} \quad x \to \pm\infty \quad (i = 1, 2; \quad j = 1, 2, 3), \tag{5.3}$$

$$\psi^s_{i+} \sim \begin{Bmatrix} (e^{iKx} + r_{i+} e^{-iKx}) e^{Ky} & \text{as} & x \to +\infty \\ t_{i+} e^{iKx+Ky} & \text{as} & x \to -\infty \end{Bmatrix} \quad (i = 1, 2), \tag{5.4}$$

$$\psi^s_{i-} \sim \begin{Bmatrix} t_{i-} e^{-iKx+Ky} & \text{as} & x \to +\infty \\ (e^{-iKx} + r_{i-} e^{iKx}) e^{Ky} & \text{as} & x \to -\infty \end{Bmatrix} \quad (i = 1, 2). \tag{5.5}$$

With these solutions to the single-body problems we are now able to study the radiation and scattering problems for two bodies as formulated in § 2. We assume that the first cylinder is a distance b from the origin along the $+x$ axis and that the second cylinder is a distance b' from the origin along the $-x$ axis ($b, b' > 0$; see figure 1). Wide spacing implies that $b + b' \gg \lambda = 2\pi K^{-1}$. First we consider the radiation problems and then the scattering problem.

5.1. *The radiation problem for* ϕ_{1j}

Here we consider the cylinder at $x = -b'$ to be fixed, with no incident wave, and the cylinder at $x = b$ to be moving in the jth mode. Near the cylinder at $x = b$ we may write

$$\phi_{1j} = \psi^r_{1j}(x-b, y) + \epsilon_1 \psi^s_{1-}(x-b, y), \tag{5.6}$$

and near the cylinder at $x = -b'$ we may write

$$\phi_{1j} = \eta_1 \psi^s_{2+}(x+b', y), \tag{5.7}$$

where ϵ_1 and η_1 are unknown, complex constants. We have $x-b$ and $x+b'$ in (5.6) and (5.7), rather than x, since the cylinders are not situated at the origin, whereas the solutions for a single cylinder refer to the origin. Note that the first term on the right-hand side of (5.6) is that part of the solution due to the movement of the cylinder at $x = b$ in the absence of the cylinder at $x = -b'$. The second term represents the interaction due to the scattering by the cylinder at $x = b$ of the wave (of unknown potential amplitude ϵ_1) which is reflected back from the cylinder at $x = -b'$. In (5.7) the right-hand side represents the wave (of unknown potential amplitude η_1) scattered by the cylinder at $x = -b'$.

To determine ϵ_1 and η_1 we assume that there exists a region between the cylinders where the asymptotic forms (5.3)–(5.5) apply. If we substitute these forms into (5.6) and (5.7) and equate like exponentials, we obtain

$$\epsilon_1 = a^-_{1j} r_{2+} \{\exp[2iK(b+b')] - r_{2+} r_-\}^{-1}, \tag{5.8}$$

$$\eta_1 = a^-_{1j} \exp[iK(b+b')] \{\exp[2iK(b+b')] - r_{2+} r_{1-}\}^{-1}. \tag{5.9}$$

Now from (2.10) and (5.3)–(5.7) we obtain

$$A_{1j}^{+} = a_{1j}^{+} \exp(iKb) + \epsilon_1 t_{1-} \exp(iKb), \tag{5.10}$$

$$A_{1j}^{-} = \eta_1 t_{2+} \exp(iKb'). \tag{5.11}$$

From (A 17) we know that

$$\omega^2 M_{1j1j} - i\omega B_{1j1j} = -\rho\omega^2 \int_{S_1} \phi_{1j} \frac{\partial \phi_{1j}}{\partial n}\, dl,$$

which gives, after use of (5.1), (5.2) and (5.6),

$$\begin{aligned}\omega^2 M_{1j1j} - i\omega B_{1j1j} &= -\rho\omega^2 \int_{S_1} (\psi_{1j}^{r}(x-b,y) + \epsilon_1 \psi_{1-}^{s}(x-b,y)) \frac{\partial \psi_{1j}^{r}}{\partial n}\, dl \\ &= \omega^2 m_{jj}^{(1)} - i\omega b_{jj}^{(1)} - \rho\omega^2 \epsilon_1 \int_{S_1} \psi_{1-}^{s}(x-b,y) \frac{\partial \psi_{1j}^{r}}{\partial n}\, dl, \end{aligned} \tag{5.12}$$

where $m_{jj}^{(1)}$ and $b_{jj}^{(1)}$ are the added-mass and damping coefficients for the cylinder at b in isolation. By use of Green's theorem (A 5) it is possible to show that

$$\int_{S_1} \psi_{1-}^{s}(x-b,y) \frac{\partial \psi_{1j}^{r}}{\partial n}\, dl = -i a_{1j}^{-},$$

so that from (5.12)

$$\omega^2 M_{1j1j} - i\omega B_{1j1j} = \omega^2 m_{jj}^{(1)} - i\omega b_{jj}^{(1)} + i\rho\omega^2 \epsilon_1 a_{1j}^{-}. \tag{5.13}$$

5.2. *Radiation problem for ϕ_{2k}*

The analysis proceeds as for ϕ_{1j}. Near the cylinder at $x = b$

$$\phi_{2k} = \eta_2 \psi_{1-}^{s}(x-b,y), \tag{5.14}$$

and near the cylinder at $x = -b'$

$$\phi_{2k} = \psi_{2k}^{r}(x+b',y) + \epsilon_2 \psi_{2+}^{s}(x+b',y), \tag{5.15}$$

where ϵ_2 and η_2 are complex constants. Matching, as in § 5.1, gives

$$\epsilon_2 = a_{2k}^{+} r_{1-} \{\exp[2iK(b+b')] - r_{2+} r_{1-}\}^{-1}, \tag{5.16}$$

$$\eta_2 = a_{2k}^{+} \exp[iK(b+b')]\{\exp[2iK(b+b')] - r_{2+} r_{1-}\}^{-1}. \tag{5.17}$$

Hence, from (2.10), (5.14) and (5.15), we obtain

$$A_{2k}^{+} = \eta_2 t_{1-} \exp(iKb), \tag{5.18}$$

$$A_{2k}^{-} = a_{2k}^{-} \exp(iKb') + \epsilon_2 t_{2+} \exp(iKb'), \tag{5.19}$$

$$\omega^2 M_{2k2k} - i\omega B_{2k2k} = \omega^2 m_{kk}^{(2)} - i\omega b_{kk}^{(2)} + i\rho\omega^2 \epsilon_2 a_{2k}^{+}, \tag{5.20}$$

where $m_{kk}^{(2)}$ and $b_{kk}^{(2)}$ are the added-mass and damping coefficients for the cylinder at $-b'$ in isolation.

With the above information it is possible to calculate M_{1j2k}, B_{1j2k}, M_{2k1j} and B_{2k1j}. From (A 17)

$$\omega^2 M_{1j2k} - i\omega B_{1j2k} = -\rho\omega^2 \int_{S_1} \phi_{2k} \frac{\partial \phi_{1j}}{\partial n}\, dl.$$

We substitute for ϕ_{1j} from (5.6) and for ϕ_{2k} from (5.14) to obtain

$$\begin{aligned}\omega^2 M_{1j2k} - i\omega B_{1j2k} &= -\rho\omega^2 \int_{S_1} \eta_2 \psi_{1-}^{s}(x-b,y) \frac{\partial \psi_{1j}^{r}}{\partial n}\, dl \\ &= i\rho\omega^2 \eta_2 a_{1j}^{-}, \end{aligned} \tag{5.21}$$

on using Green's theorem. Similarly

$$\omega^2 M_{2k1j} - i\omega B_{2k1j} = i\rho\omega^2 \eta_1 a_{2k}^{+}. \tag{5.22}$$

Note that (5.21) and (5.22) together with (5.9) and (5.17) check with result (A 18).

5.3. *Scattering problem for ϕ_s*

In this case both cylinders are held fixed and a wave is incident upon them from $x = +\infty$. As above we find, near the cylinder at $x = b$

$$\phi_s = \exp(iKb)\, \psi_{1+}^{s}(x-b, y) + \epsilon_3 \psi_{1-}^{s}(x-b, y), \tag{5.23}$$

and near the cylinder at $x = -b'$

$$\phi_s = \eta_3 \psi_{2+}^{s}(x+b', y), \tag{5.24}$$

where ϵ_3 and η_3 are complex constants. Note that the factor $\exp(iKb)$ is necessary in the first term on the right-hand side of (5.23) to give the right behaviour for the incident wave as $x \to +\infty$, consistent with (2.8). Again by matching (5.23) to (5.24) we obtain

$$\epsilon_3 = t_{1+} r_{1+} \exp(iKb) \{\exp[2iK(b+b')] - r_{2+} r_{2-}\}^{-1}, \tag{5.25}$$

$$\eta_3 = t_{1+} \exp[iK(2b+b')] \{\exp[2iK(b+b')] - r_{2+} r_{1-}\}^{-1}. \tag{5.26}$$

From (2.8), (2.9), (5.23) and (5.24) we obtain

$$R = r_{1+} \exp(2iKb) + \epsilon_3 t_{1-} \exp(iKb) \tag{5.27}$$

$$T = \eta_3 t_{2+} \exp(iKb'). \tag{5.28}$$

We have now derived an approximate solution to the two-cylinder problem in terms of solutions to the single-cylinder problems. There are many numerical solutions to the radiation and scattering problems involving single cylinders of different cross-sections. In contrast there are a limited number of analytic solutions. Perhaps the best known of these are the exact solutions derived by Ursell (1947, 1948) for a thin, partly immersed, vertical barrier. In the next section we exploit these solutions to consider wave absorption by two such thin, vertical, rolling barriers.

It would be of interest to check whether our approximate solutions for the damping coefficients B_{ijpk} and for $A_{ij}^{\pm}$, R and T satisfy (A 7), (A 19) and (A 20). This, however, is a difficult exercise as $a_{ij}^{\pm}$, $r_{i\pm}$, $t_{i\pm}$ and $b_{jk}^{(i)}$ themselves satisfy similar relations (given below). This means that it is not obvious how to use these relations to simplify the expressions derived for the two-cylinder case. However (A 7), (A 19) and (A 20) have been verified numerically for the special case considered in the next section.

Note that from Newman [1975, equation (13); 1976, equations (31*a*), (38)–(42)] we know that for $i = 1, 2$ and $j = 1, 2, 3$

$$\overline{a_{ij}^{+}} r_{i+} + \overline{a_{ij}^{-}} t_{i+} + a_{ij}^{+} = 0,$$

$$\overline{a_{ij}^{-}} r_{i-} + \overline{a_{ij}^{+}} t_{i-} + a_{ij}^{-} = 0,$$

$$|r_{i\pm}|^2 + |t_{i\pm}|^2 = 0, \quad t_{i+} \overline{r_{i-}} + \overline{t_{i-}} r_{i+} = 0,$$

$$t_{i+} = t_{i-}, \quad |r_{i+}| = |r_{i-}|,$$

$$\arg(r_{i+}) + \arg(r_{i-}) = \pi + 2\arg(t_{i+}),$$

$$b_{jk}^{(i)} = \tfrac{1}{2}\rho\omega\{a_{ij}^{+}\overline{a_{ik}^{+}} + a_{ij}^{-}\overline{a_{ik}^{-}}\} = \tfrac{1}{2}\rho\omega\{\overline{a_{ij}^{+}} a_{ik}^{+} + \overline{a_{ij}^{-}} a_{ik}^{-}\}.$$

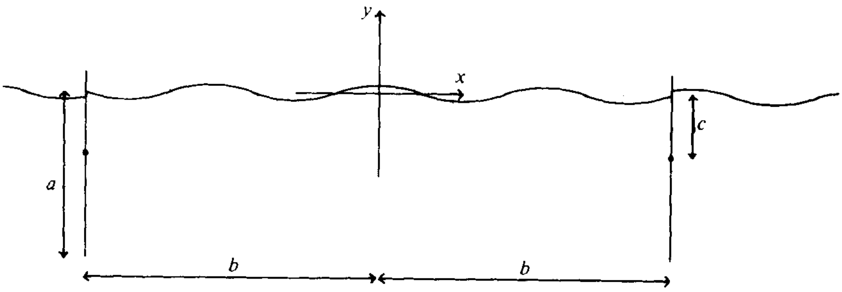


FIGURE 2. Configuration of two thin vertical barriers acting as wave-power absorbers.

6. Wave-energy absorption by two rolling vertical barriers

In this section we consider the special case of two identical rolling vertical thin barriers a distance $2b$ apart, positioned symmetrically with respect to the origin and each immersed to a depth a. The barriers can roll about a point a distance c along their length in response to the incident wave, the energy being absorbed through a linear spring–damper system attached to each barrier. The configuration is shown in figure 2.

Results for the efficiency of a single barrier rolling about a point in the free surface have been given by Evans (1976), where the maximum efficiency was shown to be 50 %. As in that paper we exploit the known explicit solutions derived by Ursell for the velocity potential associated with the scattering of a given incident wave from a single thin vertical barrier (1947) and also for the waves produced by the rolling of such a barrier (1948). Thus, in the notation of § 5, for $i = 1, 2$,

$$a_{i3}^{+} = -a_{i3}^{-}, \tag{6.1}$$

$$|a_{i3}^{+}|/a^2 = \pi(Ka)^{-1}(\pi^2 I_1^2 + K_1^2)^{-\frac{1}{2}}\left|\tfrac{1}{2} - (Ka)^{-1}(1-Kc)(I_1+L_1)\right|, \tag{6.2}$$

$$\arg(a_{i3}^{+}) = \tan^{-1}(K_1/\pi I_1) + \tfrac{1}{2}\pi, \tag{6.3}$$

$$r_{i\pm} = \pi I_1/(\pi I_1 - iK_1), \quad t_{i\pm} = -iK_1/(\pi I_1 - iK_1), \qquad (6.4), (6.5)$$

where K_1 and I_1 are modified Bessel functions and L_1 a modified Struve function, all with argument Ka.

The added-mass and damping coefficients for a single rolling vertical plate have recently been derived explicitly from Ursell's theory (1948) by Mei (1976). Thus for $i = 1, 2$, we have from Mei (1976), after correction of a typographical error,

$$b_{33}^{(i)}/(\tfrac{1}{4}\pi\rho\omega a^4) = 4\pi(Ka)^{-2}(\pi^2 I_1^2 + K_1^2)^{-1}\left[\tfrac{1}{2} - (Ka)^{-1}(1-Kc)(I_1+L_1)\right]^2 \tag{6.6}$$

$$\begin{aligned} m_{33}^{(i)}/(\tfrac{1}{4}\pi\rho a^4) = 4(Ka)^{-4} I_1^{-1}\Big\{ & K_1(\pi^2 I_1^2 + K_1^2)^{-1}\left[\tfrac{1}{2}Ka - (1-Kc)(I_1+L_1)\right]^2 \\ & - \tfrac{1}{4}(Ka)^3 I_0 + Ka[(1-Kc)L_1 - Ka][I_1 L_0 - I_0 L_1](1-Kc) \\ & - 2\pi^{-1} I_1 (1-Kc)^2 \int_0^{Ka} x L_1(x)\,dx \\ & + (Ka)^2 I_1\left[\tfrac{1}{2} + \tfrac{2}{3}\pi^{-1}(1-Kc)Ka + \tfrac{1}{16}(Ka)^2\right]\Big\}, \end{aligned} \tag{6.7}$$

where K_1, I_1 and L_1 are as in (6.2) and I_0 and L_0 are modified Bessel and Struve functions respectively, with argument Ka.

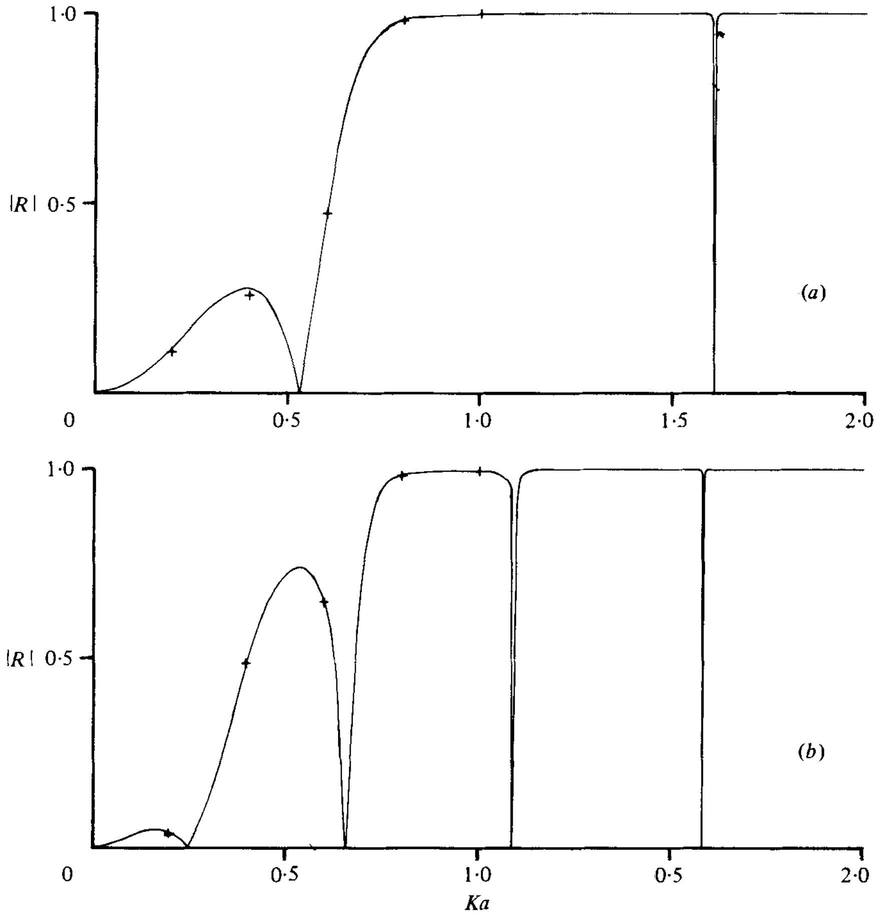


FIGURE 3. Comparison of results for $|R|$ with Evans & Morris. (*a*) $b/a = 1$. (*b*) $b/a = 3$. +, results taken from Evans & Morris; ——, present results.

The results (6.1)–(6.7) provide all the information we require in order to predict the efficiency of a system of two independently rolling thin vertical plates as a wave-power absorber. All that is required is to substitute (6.1)–(6.7) into the general expression for the efficiency given by (4.16). The amount of computation involved is reduced considerably by exploiting the symmetry of the two-barrier system. Thus

$$A^{\pm}_{13} = -A^{\mp}_{23}, \quad B_{1313} = B_{2323}, \quad M_{1313} = M_{2323}, \tag{6.8}$$

and from (A 18)

$$B_{1323} = B_{2312}, \quad M_{1323} = M_{2313}. \tag{6.9}$$

7. Results and discussion

As a check on the accuracy of the wide-spacing approximation of § 5 a comparison was made with the results of Evans & Morris (1972) for the scattering of surface waves by two vertical barriers. They made use of two different variational approximations to find the reflexion coefficient R. Their approximations were found to be good provided that the spacing between the two barriers was not too small. They gave curves of $|R|$ against Ka for $b/a = 1, 3$ and we have computed $|R|$ for the same values of b/a

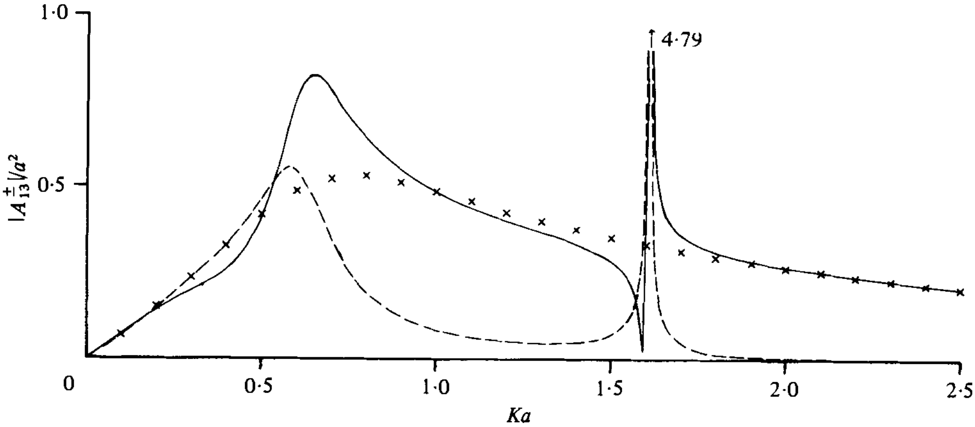


FIGURE 4. Non-dimensional potential amplitudes at $x = \pm\infty$ for one barrier rolling in the presence of another ($b/a = 1$, $c/a = 0$). ——, $|A_{13}^{+}|/a^2$, potential amplitude at $x = +\infty$; - - -, $|A_{13}^{-}|/a^2$, potential amplitude at $x = -\infty$; ×, $|a_{13}^{\pm}|/a^2$, non-dimensional potential amplitudes at $x = \pm\infty$ for a single rolling barrier ($c/a = 0$).

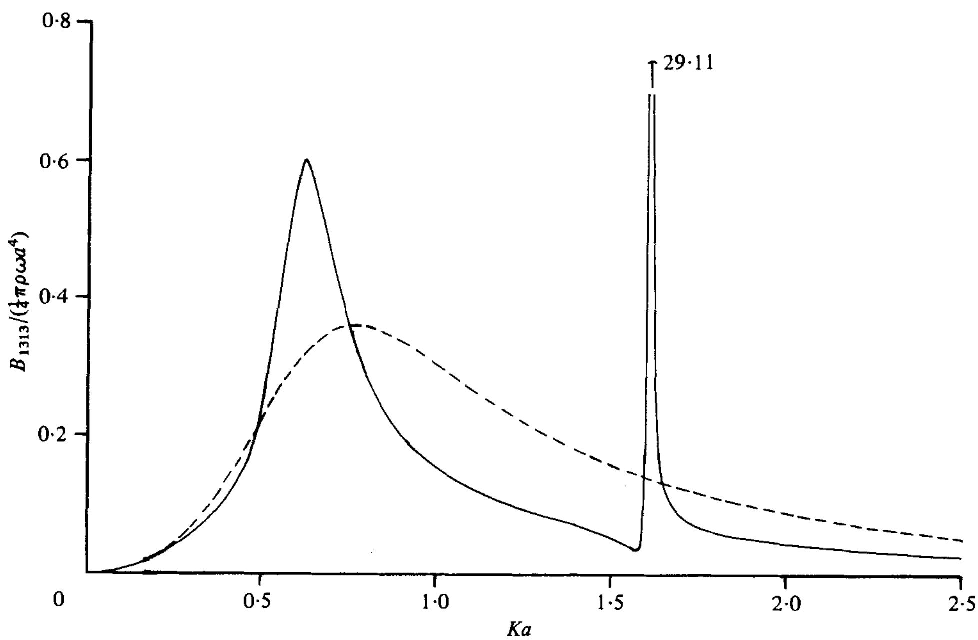


FIGURE 5. Non-dimensional damping coefficient for one barrier rolling in the presence of another ($b/a = 1, c/a = 0$). ——, $B_{1313}/(\frac{1}{4}\pi\rho\omega a^4)$; - - -, $b_{13}^{(1)}/(\frac{1}{4}\pi\rho\omega a^4)$, non-dimensional damping coefficient for a single rolling barrier ($c/a = 0$).

(see figure 3). As can be seen from figure 3 (b), for $b/a = 3$, no distinction can be made between the values of $|R|$ given by the wide-spacing approximation and those given by Evans & Morris. From figure 3 (a) it is clear that, although the wide-spacing approximation deviates slightly from the results of Evans & Morris, it is still good. If we consider, for example, the value of $|R|$ at $Ka = 0{\cdot}6$ we find that it agrees closely with the value given by Evans & Morris. Since this corresponds to the wavelength λ being

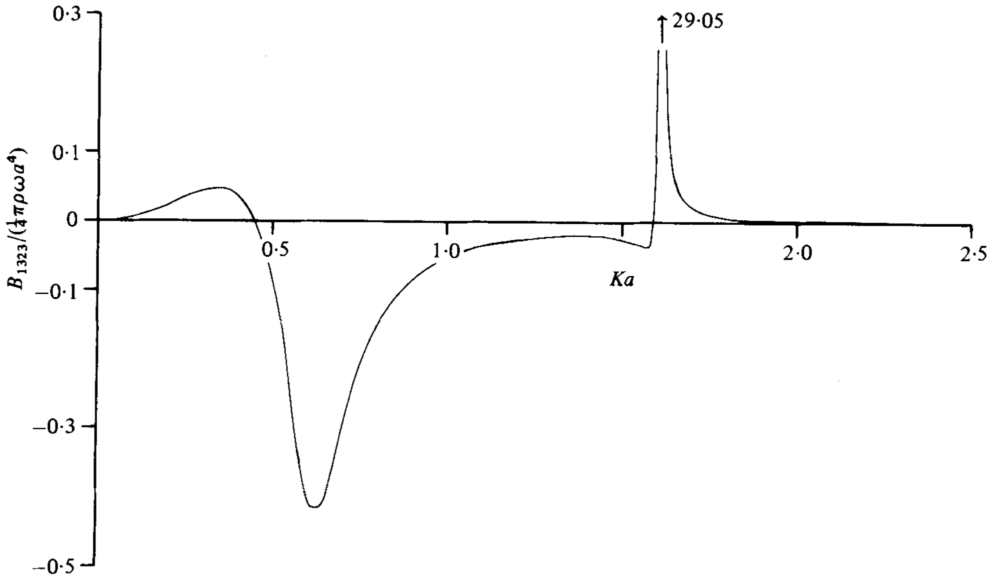


FIGURE 6. Non-dimensional damping coefficient for one barrier arising from the rolling motion at the second barrier ($b/a = 1$, $c/a = 0$).

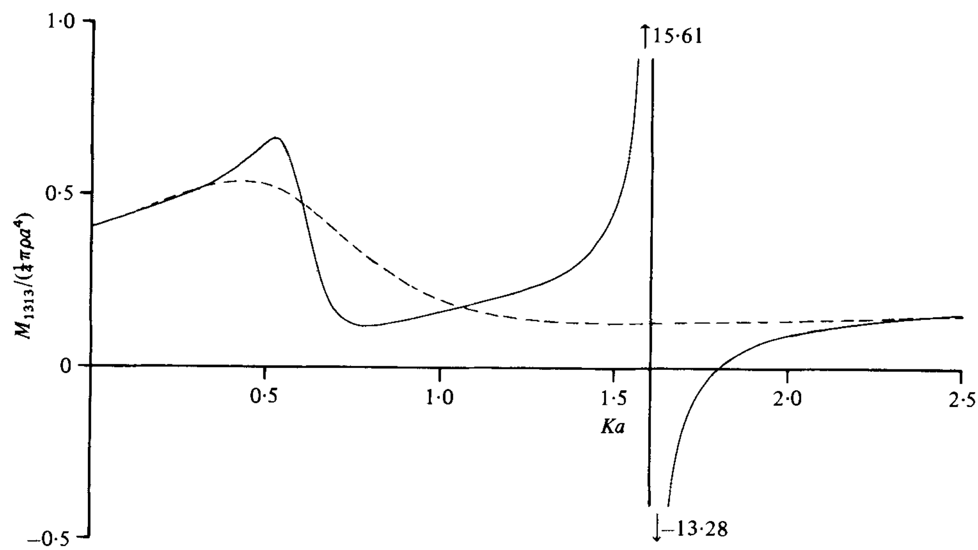


FIGURE 7. Non-dimensional added-mass coefficient for one barrier rolling in the presence of another ($b/a = 1$, $c/a = 0$). ——, $M_{1313}/(\frac{1}{4}\pi\rho a^4)$; − − −, $m_{13}^{(1)}/(\frac{1}{4}\pi\rho a^4)$, non-dimensional added-mass coefficient for a single rolling barrier ($c/a = 0$).

five times the barrier spacing, it is clear that the approximation can be good when the initial assumption of wide barrier spacing ($\lambda \ll 2b$) is not valid. In order to guarantee the accuracy of the method we shall restrict the computations to $b/a \geqslant 1$.

As noted previously, it was not possible to show that the approximate results of § 5 (for R, T, A_{ij} and B_{ijpk}) satisfied (A 7) and (A 19), owing to the complexity of the algebra involved. However by using the computed values of these quantities it was possible to check out these relations for a range of values of the parameters Ka, b/a

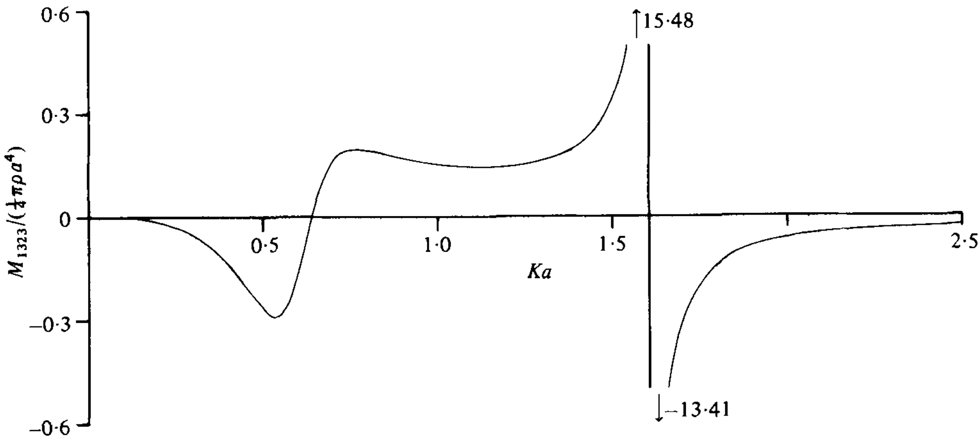


FIGURE 8. Non-dimensional added-mass coefficient for one barrier arising from the rolling motion of the second barrier ($b/a = 1$, $c/a = 0$).

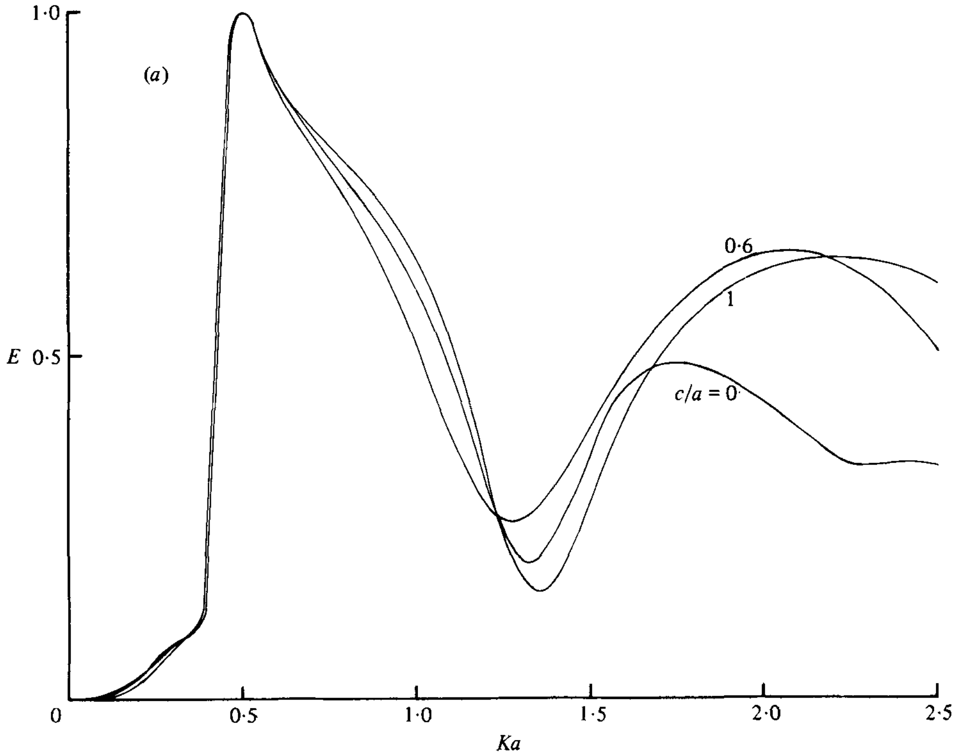
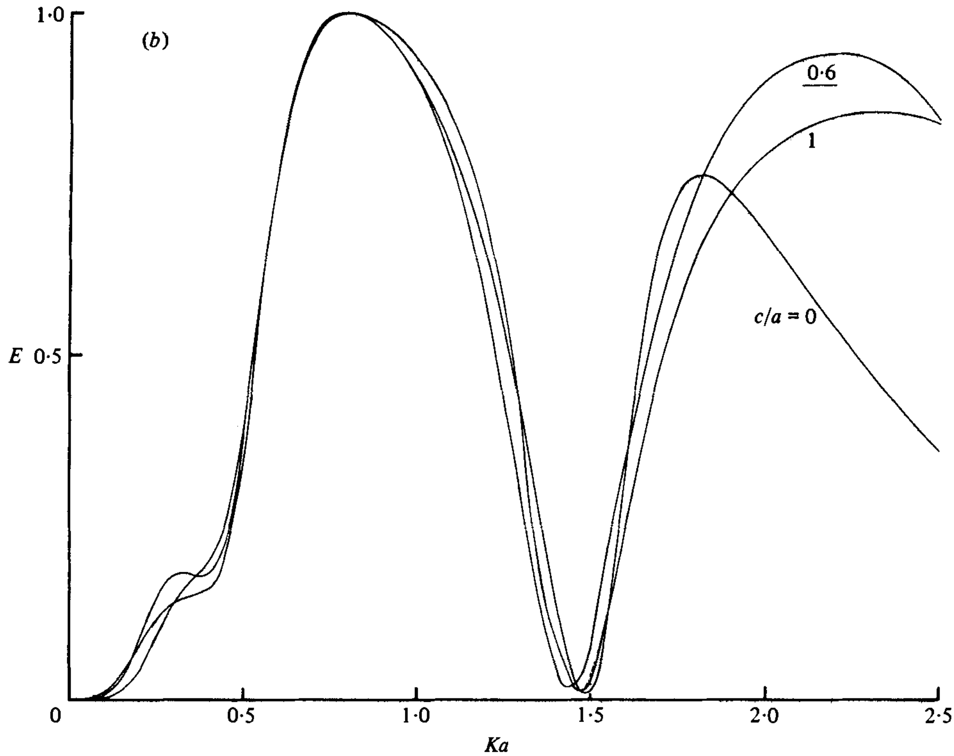


FIGURE 9a. For description see opposite.

and c/a. As these relations are satisfied by the numerical results this shows that both the approximation used and the numerical results are consistent with the general theory presented earlier.

Before proceeding to consider the efficiency E of the system as a wave-power absorber we briefly consider the results for $A_{13}^{\pm}$, M_{1313}, B_{1313}, M_{1323} and B_{1323} given in figures 4–8 in non-dimensional form. For comparison, the non-dimensional values of



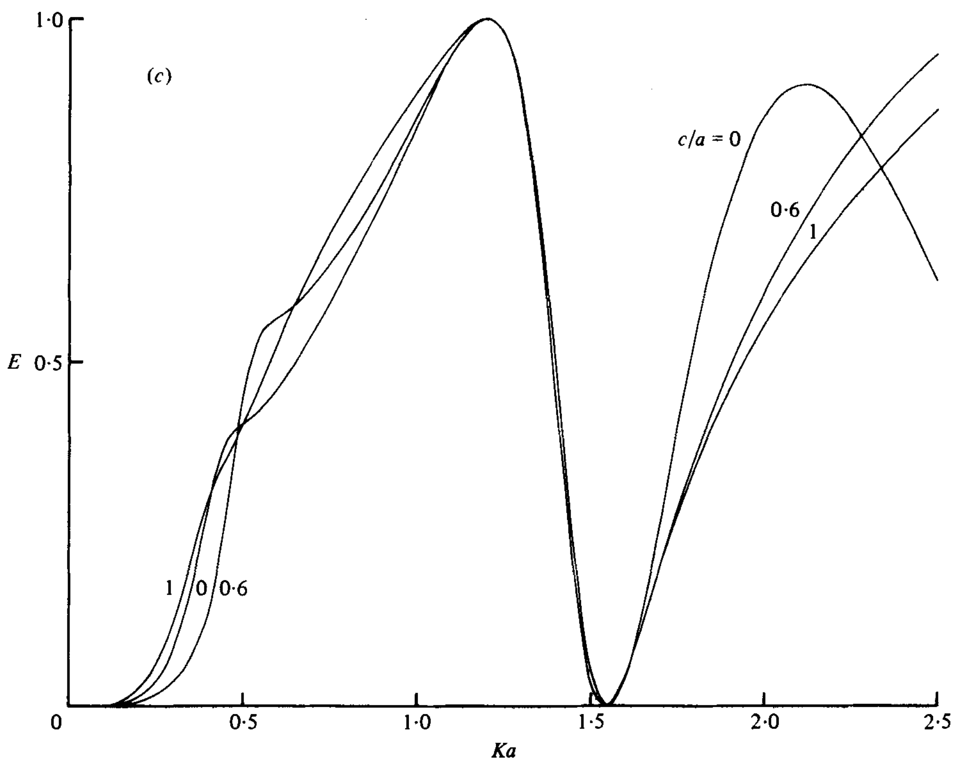


FIGURE 9. Efficiency E *vs.* non-dimensional wavenumber Ka for $b/a = 1$ and (*a*) $K_0a = 0{\cdot}5$, (*b*) $K_0a = 0{\cdot}8$, (*c*) $K_0a = 1{\cdot}2$.

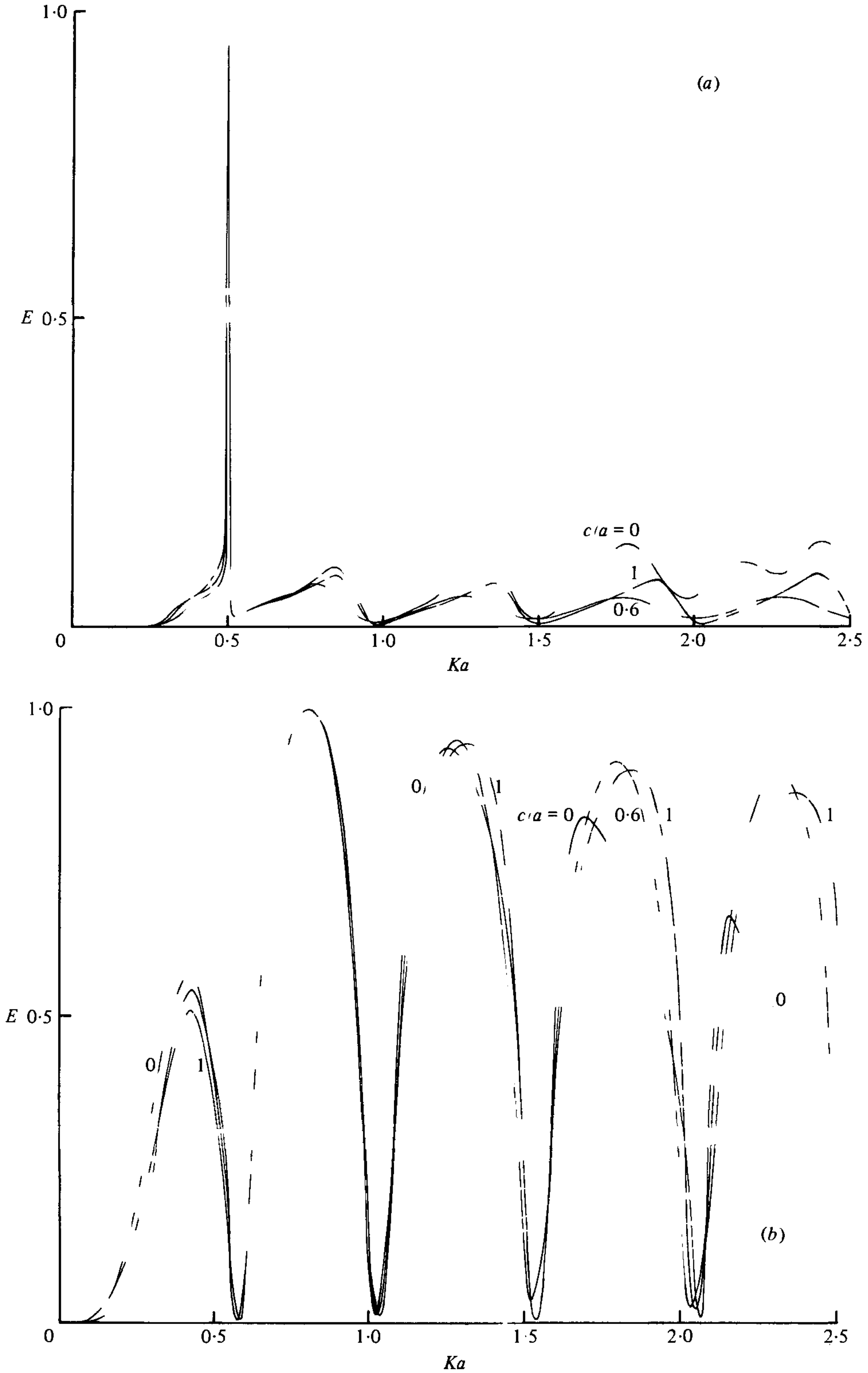


FIGURE 10 (*a*, *b*). For description see opposite.

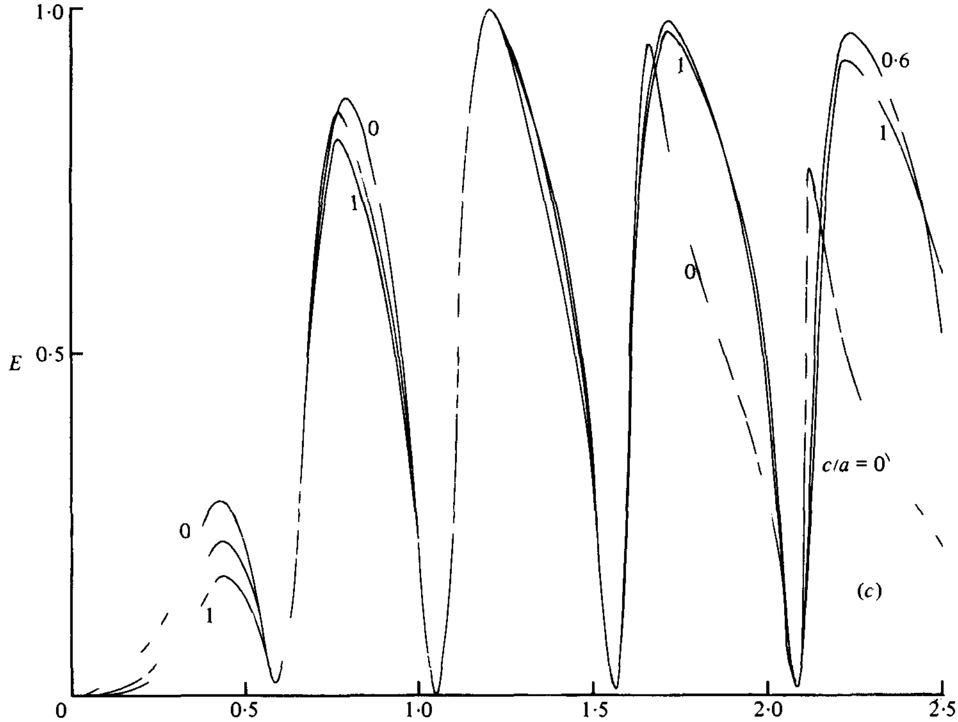


FIGURE 10. Efficiency E *vs.* non-dimensional wavenumber Ka for $b/a = 3$ and (*a*) $K_0a = 0{\cdot}5$, (*b*) K_0a $0{\cdot}8$, (*c*) $K_0a = 1{\cdot}2$.

the added-inertia and damping coefficients and the potential amplitude for a single barrier are shown in figures 7, 5 and 4 respectively.

It is interesting to note that there exists a discrete set of characteristic frequencies at which the motion of the fluid between the two plates is strongly excited by the oscillations of one plate with the other held fixed. These characteristic frequencies are given approximately by the equation

$$2Kb = q\pi \quad (q = 1, 2, \ldots), \tag{7.1}$$

which relates to the natural modes of motion of the fluid between two vertical plates extending to $y = -\infty$ and a distance $2b$ apart with no energy dissipation. In the case we are considering, as the plates are finite there is always a leakage of energy under the plates, which is then transmitted to infinity. Consequently peak resonant frequencies are not exactly coincident with those specified by (7.1). Similar resonances were found to occur owing to the oscillations of twin circular cylinders in the free surface, by Wang & Wahab (1971). Newman (1977) found resonant effects in a similar three-dimensional problem.

For the results in the particular case $b/a = 1$, $c/a = 0$ (shown in figures 4–8), (7.1) gives $Ka = \frac{1}{2}\pi$ (for $q = 1$) as the first value of Ka near which resonance occurs. This is clearly confirmed by the results given in figures 4–8. For $q \geqslant 2$ the resonant effects are outside the range of Ka considered. Calculations (not shown) were carried out for the case $b/a = 3$, $c/a = 0$ with Ka ranging from 0 to $2{\cdot}5$. As expected, resonances were again found near the values of $Ka\,(=\frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi)$ predicted by (7.1). These resonant effects have a crucial effect on the efficiency E of the system and a more detailed discussion of them is given by Wang & Wahab (1971).

Curves of the efficiency of wave absorption E for the two barriers are shown in figures 9 and 10. The inertia of the two thin barriers will be negligible compared with the added inertia due to the fluid, so we put $m_i = 0$ $(i = 1, 2)$ in the expression (4.16) for E. The two-barrier system was tuned to different wave frequencies by choosing the spring and damper constants accordingly. Computations of E were made for barrier spacing ratios $b/a = 1$, 3, points of rotation $c/a = 0$, 0·6, 1, non-dimensional tuned wavenumbers $K_0a = 0{\cdot}5$, 0·8, 1·2 and non-dimensional wavenumbers Ka ranging from 0 to 2·5.

In figure 9(*a*) the efficiency E of the two plates is plotted against Ka for the case in which the system is tuned to give $E_{\max} = 1$ at $K_0a = 0{\cdot}5$. From the curves it can be seen that E is greater than $\frac{1}{2}$ for a significant range of values of Ka, thus improving on the single-barrier results given in Evans (1976). By varying the depth c of the point of rotation it is possible to widen the range of Ka for which $E > \frac{1}{2}$. Values of E (not shown) for $c/a = 0{\cdot}2$, 0·4 and 0·8 were also computed, but no significant improvement in E was found; in fact, in general the values of E for these values of c/a were lower than the ones shown in figure 9(*a*). However for all values of c/a a sharp decrease in the value of E was found in the neighbourhood of $Ka = 1{\cdot}3$. This decrease could not be eliminated by varying the value of c/a.

In figures 9(*b*) and (*c*) we have plotted E against Ka for $K_0a = 0{\cdot}8$ and 1·2 respectively. In both cases the range of Ka for which $E > \frac{1}{2}$ is wider than that for the case $K_0a = 0{\cdot}5$. As in figure 9(*a*), curves are shown only for those values of c/a which give the highest efficiencies. Again it is important to note that, for both $K_0a = 0{\cdot}8$ and $K_0a = 1{\cdot}2$, E decreases to very low values in the neighbourhood of $Ka = 1{\cdot}5$.

The sudden decreases in the value of E occur in all cases and are due to the resonances present in the values of the potential amplitude and added-inertia and damping coefficients. Thus in (4.11)–(4.14) values of d_i and k_i $(i = 1, 2)$ were chosen such that E attained its maximum at a given frequency ω_0. These tuned values of d_i and k_i $(i = 1, 2)$ depend on the values of the potential amplitudes and added-inertia and damping coefficients at $\omega = \omega_0$. These hydrodynamic coefficients vary most rapidly with ω near resonance. Since E, given by (4.16), depends on these coefficients, we should expect the greatest decrease in the efficiency E in the neighbourhood of the resonant frequencies. By comparing the values of Ka for which resonance occurs (figures 4–8) with the values at Ka for which E is small (figure 9), we see that this is increasingly true as K_0a approaches Ka. Thus the possibility of resonant interactions between the barriers and the fluid clearly has a detrimental effect on the efficiency of the system.

A similar effect is evident in the values of the efficiency E, plotted against Ka, for $b/a = 3$ (figure 10). Here the tuned non-dimensional wavenumbers are $K_0a = 0{\cdot}5$, 0·8 and 1·2 in figures 10(*a*), (*b*) and (*c*) respectively. Again we find sudden decreases in the efficiency E near the values of $Ka\,(=\frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi)$ given by (7.1), close to which resonances occur. As the number of these resonances is greater for $b/a = 3$ than for $b/a = 1$ in the range of Ka considered ($Ka = 0$–2·5), so is the number of sudden decreases in E. This suggests that barriers closer together are more efficient as wave-power absorbers.

The results for E shown in figure 10(*a*) are particularly poor, with $E < 0{\cdot}15$ for practically all values of Ka considered. This is because $K_0 a = 0{\cdot}5$, and the system is tuned to give $E_{\max} = 1$ in the neighbourhood of the first resonance, near $Ka = \frac{1}{8}\pi$. In figures 10(*b*) and (*c*), $K_0 a = 0{\cdot}8$ and $1{\cdot}2$ respectively so the system is not tuned to (or near) a resonant frequency. This results in a considerable improvement in the efficiency of the system. As in the case $b/a = 1$, the values of E can be improved by varying c/a. However it is not possible to eliminate the low values of E which occur owing to resonant effects.

8. Conclusion

A theory has been developed for predicting the efficiency of energy absorption of two independently oscillating cylinders. The efficiency was shown to depend upon generalized added-mass and damping coefficients as well as radiated potential amplitudes. A simple approximate method for deriving these hydrodynamic coefficients from knowledge of corresponding coefficients for a single cylinder was presented, based on the assumption that the cylinders were widely spaced. A check on the results suggested that the theory could also be used when this assumption was not satisfied.

The results show that it is possible to absorb all the energy in a given incident wave at a given frequency $\omega = \omega_0$ when the system is tuned to that frequency. They also show that, away from the tuned frequency ω_0, the efficiency of the system will be low in the neighbourhood of values of ω for which resonance occurs. Clearly from these results (and those of Wang & Wahab 1971) we can expect similar effects to occur whenever two long cylinders, of arbitrary cross-section, lying in parallel in the free surface are used to absorb energy from the incident wave. This will be so because of the possibility of resonant interaction between the two cylinders and the fluid whenever both the cylinders intersect the free surface. It is possible to eliminate resonant effects from the range of values of Ka in which we are interested by placing the barriers closer together. For example if we are interested in $\lambda_0 < \lambda < \lambda_1$ $(\lambda = 2\pi K^{-1})$, we can choose $2b < \frac{1}{2}\lambda_0$, which will ensure that no resonances occur in the range $\lambda_0 < \lambda < \lambda_1$. Another possible way of reducing resonance effects is to submerge one or both of the cylinders. This last possibility is suggested by the results of Wang (1970) on the oscillations of twin-hulled submerged cylinders, where it is shown that the more deeply submerged the two cylinders are the less pronounced are the resonant peaks in the hydrodynamic coefficients.

Detailed calculations have been made for a pair of vertical rolling plates. Such a configuration is attractive mathematically since the hydrodynamic coefficients for a single plate are known in closed form, whereas for any other shape only numerical results are available. It is not anticipated that the results would be markedly changed for bodies of different shapes; the drop in efficiency due to resonant effects would still occur but at slightly different values of Ka. The inertia of a 'full' body would narrow the efficiency curves but would have no effect on the peak efficiencies, whereas the built-in buoyancy of such a body would prevent tuning in heave at low frequencies. Both these effects are described by Evans (1976) for the case of a single body.

It is of interest to note that a system of vertical plates has been suggested recently as a possible practical wave-power machine by Farley, Parks & Altmann (1978).

The present theory deals only with two-dimensional cylinders whereas any practical wave-energy device is necessarily three-dimensional. However it is hoped to extend the present ideas to cover systems of three-dimensional bodies, where it is anticipated that resonant effects will be less important.

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Appendix A. Derivation of special relations for a system of N bodies situated on or beneath the free surface

In this section we extend the theory of § 2 to the case of N cylinders, each assumed to be long, with horizontal generators lying parallel to the generators of the other cylinders. The fluid motion is again taken to be two-dimensional. We use the notation of § 2 for the scattering and radiation problems, but now for ϕ_{ij}, i ranges over the values 1 to N, rather than 1 and 2.

Green's theorem

Let ϕ and ψ be two complex-valued time-independent potentials such that $\operatorname{Re}\{\phi\, e^{i\omega t}\}$ and $\operatorname{Re}\{\psi e^{i\omega t}\}$ satisfy (2.1) and (2.2), and

$$\partial\phi/\partial n = \partial\psi/\partial n = 0 \quad \text{on} \quad S_i \quad (i = 1, \ldots, N), \tag{A 1}$$

where S_i is the wetted surface of the ith body,

$$\phi \sim \begin{cases} (A_1\, e^{-iKx} + A_2\, e^{+iKx})\, e^{Ky} & \text{as} \quad x \to +\infty, \\ (A_3\, e^{-iKx} + A_4\, e^{+iKx})\, e^{Ky} & \text{as} \quad x \to -\infty, \end{cases} \tag{A 2}$$

$$\psi \sim \begin{cases} (B_1\, e^{-iKx} + B_2\, e^{iKx})\, e^{Ky} & \text{as} \quad x \to +\infty, \\ (B_3\, e^{-iKx} + B_4\, e^{iKx})\, e^{Ky} & \text{as} \quad x \to -\infty, \end{cases} \tag{A 3}$$

where A_i and B_i are constants ($i = 1, 2, 3, 4$), and

$$\nabla\phi, \nabla\psi \to 0 \quad \text{as} \quad y \to -\infty. \tag{A 4}$$

By Green's theorem we know that, for any sufficiently smooth harmonic functions ϕ and ψ,

$$I(\phi, \psi) \equiv \int_C \left\{\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right\} dl = 0, \tag{A 5}$$

where C is the contour consisting of the free surface, the body surfaces S_i, the fluid bottom (at $y = -\infty$) and two vertical closures at $x = \pm X$, where $X > 0$. As ϕ and ψ satisfy the free-surface condition, (A 1) and (A 4), the only contributions to the integral come from the vertical closures at $x = \pm X$. Hence using (A 2) and (A 3) gives, as $X \to \infty$,

$$-A_1 B_2 + A_2 B_1 + A_3 B_4 - A_4 B_3 = 0. \tag{A 6}$$

Extension of the Newman relations to N bodies

If we take $\phi = \phi_{ij} - \overline{\phi_{ij}}$ and $\psi = \phi_s$ then the conditions for (A 6) to hold are satisfied since

$$\begin{aligned} \frac{\partial\phi}{\partial n_k} &= \frac{\partial\phi_{ij}}{\partial n_k} - \frac{\partial\overline{\phi}_{ij}}{\partial n_k} \quad \text{on} \quad S_k \\ &= (n_{kj} - \overline{n_{kj}})\, \delta_{ik} = 0, \end{aligned}$$

since n_{kj} is real. Now we have

$$A_1 = A^+_{ij}, \quad A_2 = -\overline{A^+_{ij}}, \quad A_3 = -\overline{A^-_{ij}}, \quad A_4 = A^-_{ij},$$

$$B_1 = R, \quad B_2 = 1, \quad B_3 = 0, \quad B_4 = T,$$

so that (A 6) gives

$$\overline{A^+_{ij}}R + \overline{A^-_{ij}}T + A^+_{ij} = 0 \quad (i = 1, 2; \;\; j = 1, 2, 3), \tag{A 7}$$

which extends the Newman (1975) relations to the case of one body oscillating in a given mode in the presence of $N-1$ fixed bodies.

Note that with ϕ as above and $\psi = \phi_{pk} - \overline{\phi}_{pk}$ we find similarly that

$$A^+_{ij}\overline{A^+_{pk}} - \overline{A^+_{ij}}A^+_{pk} + A_{ij}\overline{A^-_{pk}} - \overline{A_{ij}}A^-_{pk} = 0 \quad (i, p = 1, 2, \ldots, N; \;\; j, k = 1, 2, 3), \tag{A 8}$$

or

$$\operatorname{Im}\{A^+_{ij}\overline{A^+_{pk}} + A^-_{ij}\overline{A^-_{pk}}\} = 0.$$

Extension of the Haskind relations to N fixed bodies

The force on the ith body in the jth direction due to an incident wave of amplitude A is F^s_{ij}, and is given by

$$F^s_{ij} = -\int_{S_i} p_s n_{ij}\, dl, \tag{A 9}$$

where $p_s = \operatorname{Re}\{-i\rho g A \phi_s e^{i\omega t}\}$ and ϕ_s is written as

$$(gA/\omega)\,\phi_s = \phi_I + \phi_D, \tag{A 10}$$

where ϕ_I is the incident wave and ϕ_D the diffracted wave. Thus

$$\phi_I = gA\omega^{-1}\, e^{iKx+Ky},$$

$$\phi_D \sim A^\pm_D e^{\mp iKx+Ky} \quad \text{as} \quad x \to \pm\infty. \tag{A 11}$$

Thus, from (A 9), (A 10) and (2.7), we obtain

$$F^s_{ij} = \operatorname{Re}\left\{i\omega\rho\int_{S_i}(\phi_I + \phi_D)\,\partial\phi_{ij}/\partial n\, dl\, e^{i\omega t}\right\}. \tag{A 12}$$

From (A 5) we know that $I(\phi_{ij}, \phi_D) = 0$; in this case the only contributions to I come from the body surfaces S_k, so that

$$\sum_{k=1}^{N}\int_{S_k}(\phi_{ij}\,\partial\phi_D/\partial n - \phi_D\,\partial\phi_{ij}/\partial n)\, dl = 0.$$

Also, from (A 10) and (2.7),

$$\partial\phi_I/\partial n = -\partial\phi_D/\partial n \quad \text{on} \quad S_k \quad (k = 1, \ldots, N)$$

and

$$\partial\phi_{ij}/\partial n = 0 \quad \text{on} \quad S_k \quad (k \neq i).$$

Hence, after some manipulation, we may write (A 12) as

$$F^s_{ij} = \operatorname{Re}\left\{i\omega\rho\, e^{i\omega t}\sum_{k=1}^{N}\int_{S_k}(\phi_I\,\partial\phi_{ij}/\partial n - \phi_{ij}\,\partial\phi_I/\partial n)\, dl\right\}. \tag{A 13}$$

Equation (A 5) now gives $I(\phi_I, \phi_{ij}) = 0$ so (A 13) becomes

$$F^s_{ij} = \operatorname{Re}\left\{-i\omega\rho\, e^{i\omega t}\int_{S'}(\phi_I\,\partial\phi_{ij}/\partial n - \phi_{ij}\,\partial\phi_I/\partial n)\, dl\right\},$$

where S' consists of the vertical closures at $x = \pm X$. Hence, on using (A 11) and (2.10), we obtain, as $X \to \infty$,

$$F^s_{ij} = \mathrm{Re}\{\rho g A\, A^+_{ij} e^{i\omega t}\}, \tag{A 14}$$

an extension of Haskind's relations to the case of N bodies (Newman 1976).

Generalized added-mass and damping coefficients

Generalizing (2.5) gives

$$\phi = gA\omega^{-1}\phi_s + \sum_{p=1}^{N}\sum_{k=1}^{3} i\omega\xi_{pk}\phi_{pk} \tag{A 15}$$

as the potential for the motion of N bodies in response to an incident wave of amplitude A. The previous subsection considered the forces due to a wave incident on the fixed bodies. In this subsection the forces on the ith body in the jth direction due to the motion of all N bodies is considered. F^r_{ij}, the force on the ith body in the jth direction, is given by

$$F^r_{ij} = -\int_{S_i} p_r n_{ij}\, dl,$$

where

$$p_r = \mathrm{Re}\left\{-i\omega\rho\, e^{i\omega t} \sum_{p=1}^{N}\sum_{k=1}^{3} i\omega\xi_{pk}\phi_{pk}\right\}.$$

Hence, using (2.7),

$$\begin{aligned} F^r_{ij} &= \mathrm{Re}\left\{-\rho\omega^2 e^{i\omega t} \sum_{p=1}^{N}\sum_{k=1}^{3} \xi_{pk}\int_{S_i}\phi_{pk}\frac{\partial\phi_{ij}}{\partial n_i}\, dl\right\} \\ &= -\sum_{p=1}^{N}\sum_{k=1}^{3}(M_{ijpk}\ddot{\zeta}_{pk} + B_{ijpk}\dot{\zeta}_{pk}), \end{aligned} \tag{A 16}$$

where $\zeta_{pk}(t)$ is given by (2.6) and

$$\omega^2 M_{ijpk} - i\omega B_{ijpk} = -\rho\omega^2\int_{S_i}\phi_{pk}\frac{\partial\phi_{ij}}{\partial n}\, dl. \tag{A 17}$$

Here we define M_{ijpk} and B_{ijpk} (real) to be generalized added-mass and damping coefficients. If the subscripts i and p are dropped in (A 17) we have the definition of added mass and damping for a single body [cf. Wehausen 1971, equation (23)]. Another application of Green's theorem (A 5) now gives

$$\omega^2 M_{ijpk} - i\omega B_{ijpk} = \omega^2 M_{pkij} - i\omega B_{pkij}.$$

Hence, equating real and imaginary parts, we obtain

$$M_{ijpk} = M_{pkij}, \quad B_{ijpk} = B_{pkij}. \tag{A 18}$$

In (A 16) the term $-M_{ijpk}\ddot{\zeta}_{pk}$ gives the force on the ith body in the jth direction due to (and in phase with the acceleration of) the motion of the pth body in the kth mode. Similarly $-B_{ijpk}\dot{\zeta}_{pk}$ gives the force in phase with the velocity of the pth body in the kth mode.

A relation between energy radiation and the damping coefficient

From (A 5), it can be seen that

$$I(\overline{\phi}_{ij}, \phi_{pk}) = \overline{I(\phi_{ij}, \overline{\phi}_{pk})} = 0.$$

Therefore, using (2.2), (2.7), (2.10) and (A 17), we obtain

$$i[\overline{A^+_{ij}}A^+_{pk} + \overline{A^-_{ij}}A^-_{pk}] - (\rho\omega^2)^{-1}[\omega^2 M_{pkij} + i\omega B_{pkij} - \omega^2 M_{ijpk} + i\omega B_{ijpk}] = 0,$$

which, on use of (A 18), gives

$$B_{ijpk} = \tfrac{1}{2}\rho\omega[\overline{A^+_{ij}}A^+_{pk} + \overline{A^-_{ij}}A^-_{pk}]. \tag{A 19}$$

Furthermore, it can be seen that

$$B_{ijpk} = B_{pkij} = \tfrac{1}{2}\rho\omega[A^+_{ij}\overline{A^+_{pk}} + A^-_{ij}\overline{A^-_{pk}}], \tag{A 20}$$

which together with (A 19) confirms (A 8). These results may be compared with Newman [1962, equation (38); 1976, equation (31)]; in fact if we drop the subscripts i and p we obtain results identical with his.

Appendix B. Confirmation of the result $E_{\max} = 1$ using (4.15)

From (4.12) and (4.6) we obtain

$$\begin{aligned} d_1 &= \tfrac{1}{2}\rho\omega\{|A^+_{1j}|^2 + |A^-_{1j}|^2\} + \tfrac{1}{2}[(i\omega M_{1j2k} - B_{1j2k})\overline{A^-_{1j}}/\overline{A^-_{2k}} + (-i\omega M_{1j2k} - B_{1j2k})A^-_{1j}/A^-_{2k}] \\ &= \tfrac{1}{2}\rho\omega\{A^+_{1j}|^2 + |A^-_{ij}|^2\} + \tfrac{1}{2}|A^-_{2k}|^{-2}[i\omega M_{1j2k}(\overline{A^-_{1j}}A^-_{2k} - A^-_{1j}\overline{A^-_{2k}}) \\ &\qquad - B_{1j2k}(A^-_{1j}\overline{A^-_{2k}} + A^-_{1j}\overline{A^-_{2k}})]. \end{aligned} \tag{B 1}$$

Similarly, from (4.14) and (4.6) we obtain

$$\begin{aligned} d_2 &= \tfrac{1}{2}\rho\omega\{|A^+_{2k}|^2 + |A^-_{2k}|^2\} + \tfrac{1}{2}|A_{1j}|^{-2}[i\omega M_{1j2k}(A^-_{1j}\overline{A^-_{2k}} - \overline{A^-_{1j}}A^-_{2k}) \\ &\qquad - B_{1j2k}(A^-_{1j}\overline{A^-_{2k}} + \overline{A^-_{1j}}A^-_{2k})]. \end{aligned} \tag{B 2}$$

Hence, using (B 1), (B 2), (3.6), (3.7) and (4.15), we obtain (after some simplification)

$$\begin{aligned} E &= \gamma^{-1}[(|A^+_{1j}|^2 + |A^-_{1j}|^2)\,|A^-_{2k}|^2 - 2(A^-_{1j}\overline{A^-_{2k}} + \overline{A^-_{1j}}A^-_{2k})\,B_{1j2k}/\rho\omega \\ &\qquad + (|A^+_{2k}|^2 + |A^-_{2k}|^2)\,|A^-_{1j}|^2], \end{aligned} \tag{B 3}$$

where

$$\gamma = |\overline{A^+_{1j}}\,\overline{A^-_{2k}} - \overline{A^-_{1j}}\,\overline{A^+_{2k}}|^2. \tag{B 4}$$

Substituting for B_{1j2k} in (B 3) and using (A 19) and (A 20), we obtain

$$\begin{aligned} E &= \gamma^{-1}[(|A^+_{1j}|^2 + |A^-_{1j}|^2)\,|A^-_{2k}|^2 - A^-_{1j}\overline{A^-_{2k}}(\overline{A^+_{1j}}A^+_{2k} + \overline{A^-_{1j}}A^-_{2k}) - \overline{A^-_{1j}}A^-_{2k}(A^+_{1j}\overline{A^+_{2k}} + A^-_{1j}A^-_{2k}) \\ &\qquad + (|A^+_{2k}|^2 + |A^-_{2k}|^2)\,|A^-_{1j}|^2] \\ &= \gamma^{-1}[|A^+_{1j}|^2\,|A^-_{2k}|^2 - \overline{A^+_{1j}}A^-_{1j}\,A^+_{2k}\,\overline{A^-_{2k}} - A^+_{1j}\overline{A^-_{1j}}\,\overline{A^+_{2k}}A^-_{2k} + |A^+_{2k}|^2\,|A^-_{1j}|^2] \\ &= 1 \quad \text{from (B 4).} \end{aligned}$$

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